

# Minding one's P's & Q's : from the one loop effective action in quantum field theory to classical transport theory

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## Abstract

The one loop effective action in quantum field theory can be expressed as a quantum mechanical path integral over world lines, with internal symmetries represented by Grassmannian variables. In this paper, we develop a real time, many body, world line formalism for the one loop effective action. In particular, we study hot QCD and obtain the classical transport equations which, as Litim and Manuel have shown, reduce in the appropriate limit to the non-Abelian Boltzmann-Langevin equation first obtained by Bödeker. In the Vlasov limit, the classical kinetic equations are those that correspond to the hard thermal loop effective action. We also discuss the imaginary time world line formalism for a hot  $\phi^4$  theory, and elucidate its relation to classical transport theory.

# 1 Introduction

In classical kinetic theory, a covariant formalism can be obtained in terms of phase space averages over the trajectories of particle world lines [1]. In quantum field theory, there are many instances at finite temperature and density where classical ideas are relevant, and where a classical kinetic picture would be useful. However, it is not immediately apparent how one recovers the classical world line picture directly from quantum field theory. This is especially problematic in theories with internal symmetries.

Fortunately, in the last decade, there has been a considerable body of work relating the one loop effective action in quantum field theory to quantum mechanical path integrals over point particle Lagrangians<sup>1</sup>. For a survey of recent developments, see the review by Schubert [4]. These recent developments follow from the key insight by Berezin and Marinov [5] that internal symmetries such as color and spin had classical analogues in terms of Grassmanian variables. These obeyed classical commutation relations which, when quantized, gave the usual commutation relations for spin and color.

Brink, DiVecchia, and Howe used these Grassmanian variables to construct a classical Lagrangian for spinning world lines in an Abelian background field [6]. Subsequently, Balachandran et al. [7] and Barducci et al. [8] wrote down the following Lagrangian for a classical colored particle in a non-Abelian background field<sup>2</sup>:

$$L = -m\sqrt{\dot{x}_\mu\dot{x}^\mu} + i\lambda_a^\dagger D_{ab}\lambda_b. \quad (1)$$

Here the  $\lambda_a(\tau)$  with  $a = 1, \dots, N$  are Grassmanian dynamical variables, and  $D_{ab}\lambda_b = \dot{\lambda}_a + ig\dot{x}^\mu A_\mu^\alpha T_{ab}^\alpha \lambda_b$ . The variables  $T_{ab}^\alpha$ 's are  $N \times N$  matrices in an irreducible representation of the Lie algebra of the group. The Euler-Lagrange equations of motion are deduced in the usual way from the above Lagrangian. It was shown by Balachandran et al. and by Barducci et al. that these equations are precisely the equations written down nearly thirty years ago by S.K. Wong [9].

The connection of the work on point particle Lagrangians to quantum field theory was first made by Strassler [10]. He showed that the one loop effective action in quantum field theory could be expressed in terms of a quantum mechanical path integral over a point particle Lagrangian. For an Abelian gauge theory, Strassler showed that this Lagrangian was identical

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<sup>1</sup>The connection between fields and particles is of course relatively ancient. It goes back to the works of Feynman and Schwinger [2]. Also, Polyakov in his book [3] demonstrates the relation between particle Green functions and the n-point Feynman amplitudes of quantum field theory.

<sup>2</sup>They also considered the case of a classical colored, spinning, particle in a non-Abelian background field. We will not discuss this case here.

to the one written down previously by Brink, DiVecchia, and Howe [10]. For non-Abelian gauge theories, Strassler wrote down an expression, but it did not have a Lagrangian interpretation since the trace over color—the Wilson loop—was kept explicitly. The corresponding world line Lagrangian was first obtained by D’Hoker and Gagné using a coherent state formalism [11]. Remarkably, as observed by Pisarski and Tytgat [12], the point particle Lagrangian D’Hoker and Gagné obtained by integrating out fermions coupled to a vector background field, coincides with the Lagrangian in Eq. (1).

Pisarski and Tytgat have noted that since one can write the one loop effective action as a path integral over world lines, it is a possible explanation of the apparently mysterious result that classical kinetic theory in terms of the single particle distributions  $f(x, p, Q)$  gives rise to the well known HTL effective action for finite temperature gauge theories [12]. In this work, we extend the previous work on the vacuum world line formalism, and develop a real time many body formalism, which may be applied to a wide range of many body problems at finite temperature and density. In particular, we will focus our attention here on the apparently mysterious results that were discussed in Ref. [12], and show how they may be understood in the many body world line formalism. Before we can be more explicit, we need therefore to discuss the status of recent work in finite temperature gauge theories.

At very high temperatures, the physics of soft modes in gauge theories may be described by effective field theories [13]. Integrating out modes with momenta  $p \sim T$ , one obtains an effective theory at the scale  $p \sim gT$ , characterized by a Debye screening mass  $m_D^2 \propto g^2 T^2$ . The gauge invariant effective action describing these modes is often referred to as the Hard Thermal Loop (HTL) effective action [14, 15, 16]. Recently, there has been much progress in describing very soft magnetic modes at the scale  $p \sim g^2 T$  by systematically integrating out modes at the scales  $T$  and  $gT$  [17, 18, 19, 20, 21].

It was first shown by Blaizot and Iancu [22] that the non-local hard thermal loop effective action could be derived in a local kinetic approach by systematically truncating the Schwinger-Dyson equations. The kinetic approach was very useful since it lead to an intuitive picture in which soft modes  $p \sim gT$  satisfy classical equations of motion and are coupled to hard modes  $p \sim T$ . These hard modes in turn satisfy the collisionless Boltzmann equation.

This Schwinger-Dyson approach is a perfectly reasonable way to proceed. However, since the approximations are made at the level of the operator equations of motion, it can become cumbersome. A local effective action was developed by Iancu [23], which is more useful, and indeed has been the basis of some recent work [18, 24].

Some time later, Kelly et al. [25] wrote down an alternative transport the-

ory for classical colored particles. These particles obey the classical equations of motion of a spinless colored particle, coupled to a non-Abelian background field  $A_\mu^a$ , that were first written down by Wong [9].

In the picture of Kelly et al., the single particle distributions  $f(x, p, Q)$  are defined for an extended phase space of classical colored charges. (In this regard, their approach is similar to the earlier work of Heinz [26] and collaborators on a classical transport theory for colored charges.) They showed that this distribution obeys the collisionless Boltzmann equation. Making suitable approximations to their kinetic equations, they showed that they also recovered the HTL effective action.

The classical transport theory of Kelly et al. is very simple and elegant. However, the formalism is *ad hoc*, and it was not clear whether such a formalism could be “derived” in some systematic approximation. It was also not clear whether the distributions  $f$  were distributions in the statistical mechanics sense, since no phase space averaging had been performed in the derivation. Finally, as pointed out by Blaizot and Iancu [27], it was not clear why color could be treated as a classical degree of freedom.

The problematic issue of statistical averaging in the work of Kelly et al. was clarified recently in the work of Litim and Manuel [19]. In this work, we show how, in the real time many body world line formalism, the classical transport theory of Kelly et al., as improved by Litim and Manuel, may be obtained from the one loop effective action in QCD. We obtain the same set of transport equations they do. Since our results are derived from the QCD one loop effective action, one can, in principle, go further. We will not attempt to do so in this paper, but save that effort for later work.

This paper is organized as follows. In section 2, we summarize the world line formalism for the one loop effective action in QCD at zero temperature. In section 3, we discuss the point particle Lagrangian that results from the one loop effective action. We show that Wong’s equations follow from this Lagrangian. We comment on the classical commutation relation among the various dynamical variables and their quantum counterparts. Next, in section 4, a many body world line formalism is developed, and applied to the one loop effective action. In the following section, it is shown that the saddle point of this effective action gives the finite temperature classical transport theory of Litim and Manuel. As shown by these authors, their results agree with those obtained previously by Bödeker [17] for soft modes in hot gauge theories. The final section explores the implications of our result, and discusses further avenues of research. In appendix A, we work out, in the imaginary time approach, a simple example illustrating the finite temperature world line formalism. We also point out, in this context, the connections to classical transport theory.

## 2 The one loop effective action and the world line formalism in quantum field theory

In recent years, the world line formalism has become popular as a technique to compute one loop scattering amplitudes in gauge theories. Initial interest in such techniques followed from the work of Bern and Kosower [28], who derived rules from string theory for computing one loop scattering amplitudes in gauge theories. Strassler showed that similar rules could be obtained directly in gauge theories by writing the one loop effective action, in background field gauge, as a one dimensional quantum mechanical path integral over the “world line” of a point particle in an external background field [10].

We consider, with Strassler as a guide, the example of the Lagrangian of a massless scalar field coupled to a background Abelian gauge field  $A_\mu(x)$ ,

$$\mathcal{L} = \Phi^\dagger D^2 \Phi, \quad (2)$$

where  $D_\mu = \partial_\mu - igA_\mu$ . The one loop effective action is

$$\Gamma[A] = -\log [\det(-D^2)] \equiv -\text{Tr} (\log(-D^2)) . \quad (3)$$

(The general case of Dirac spinors coupled to scalars, vectors, axial scalars and vectors, and tensors, has been considered by several authors<sup>3</sup>.)

We then use the trick

$$\log(\sigma) = \int_1^\sigma \frac{dy}{y} \equiv \int_1^\sigma dy \int_0^\infty dt e^{-yt} = - \int_0^\infty \frac{dt}{t} (e^{-\sigma t} - e^{-t}) , \quad (4)$$

to re-write Eq. (3) as

$$\begin{aligned} \Gamma[A] &= \int_0^\infty \frac{dT}{T} \int \frac{d^4 p}{(2\pi)^4} \langle p | \exp \left( -\frac{1}{2} \varepsilon T (p + gA(x))^2 \right) | p \rangle \\ &= \int_0^\infty \frac{dT}{T} \mathcal{N} \int \mathcal{D}x \mathcal{P} \exp \left[ - \int_0^T d\tau \left( \frac{1}{2\varepsilon} \dot{x}^2 + igA[x(\tau)] \cdot \dot{x} \right) \right] \end{aligned} \quad (5)$$

where  $\mathcal{P}$  denotes the path-ordering.

Above,  $\varepsilon$  is the einbein (the square root of the one dimensional metric—taken here to be an arbitrary constant) and

$$\mathcal{N} = \int \mathcal{D}p \exp \left( -\frac{1}{2} \int_0^T d\tau \varepsilon p^2 \right) ,$$

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<sup>3</sup> for a compact review, see the talk by C. Schubert [29].

is a normalization constant. Analytically continuing this expression to Minkowski space-time and letting  $\varepsilon \longrightarrow -\varepsilon$ , one obtains in the general case of  $A_\mu$  being a matrix in the group representation  $R$  of the scalar field

$$\Gamma[A] = \int_0^\infty \frac{dT}{T} \mathcal{N} \int \mathcal{D}x \text{Tr}_R \mathcal{P} \exp \left[ i \int_0^T d\tau \left( \frac{1}{2\varepsilon} \dot{x}^2 - ig A[x(\tau)] \cdot \dot{x} \right) \right] \quad (6)$$

where  $\text{Tr}_R$  denotes the trace over the internal states. This expression can be factorized into a product of exponentials—the  $A$  dependent piece being a Wilson loop. It is therefore the expectation value of a Wilson loop of the background field, in a particular ensemble of loops.

Strassler showed that one particle irreducible (1PI) Feynman diagrams, with  $N$  external background gluons and a scalar loop, could be constructed by expanding the above expression for the one loop effective action to order  $g^N$ . Expanding the gauge field in a set of plane waves with definite momentum, polarization, and color charge, he re-wrote the above 1PI-action as factorized products of the traces over color and the  $N$  proper time integrals (and permutations thereof). He then considered several examples—and computed, for instance, the gluon vacuum polarization in a theory with Dirac fermions and complex adjoint scalars.

Our objective here is slightly different. We wish to write Eq. (6) in terms of a point particle Lagrangian of scalar particles which carry internal symmetries and are coupled to an external field. (The extension to this work to fermions is straightforward but will not be discussed here.) Towards this end, we shall here follow the later work of D'Hoker and Gagné (see also the related work of Mondragón et al. [30]). D'Hoker and Gagné proved the following identity for an  $n \times n$  Hermitean matrix  $M(\tau)$ ,

$$\begin{aligned} \text{Tr} \mathcal{P} \exp \left( i \int_0^T d\tau M(\tau) \right) &= \left( \frac{\pi}{T} \right)^n \sum_\phi \int_{AP} \mathcal{D}\lambda^\dagger \mathcal{D}\lambda e^{i\phi(\lambda^\dagger \lambda + \frac{n}{2} - 1)} \\ &\times \exp \left( - \int_0^T d\tau [\lambda^\dagger \dot{\lambda} - i\lambda^\dagger M \lambda] \right). \end{aligned} \quad (7)$$

Here  $\phi = 2\pi k/n$ , where  $k = 1, \dots, n$ . As noted in the introduction, the  $\lambda^\dagger$  and  $\lambda$  are independent dynamical Grassmanian variables (the subscript “AP” denotes anti-periodic boundary conditions) which act as eigenvalues of creation and annihilation Fermi operators, respectively, generating arbitrary finite dimensional representations of the  $SU(N_c)$  symmetry group. These are discussed further in the following section. Also, the value of  $\lambda^\dagger \lambda$  can be evaluated at any arbitrary value of  $\tau$ .

In the adjoint case, the matrix  $M$  stands for the adjoint gauge field  $A^a T^a$ , with  $n = N_c^2 - 1$ . Substituting the above identity in Eq. (6), we can write

the one loop effective action for the Lagrangian in Eq. (2) as,

$$\begin{aligned} \Gamma[A] &= \int_0^\infty \frac{dT}{T} \mathcal{N} \int \mathcal{D}x \int_{AP} \mathcal{D}\lambda^\dagger \mathcal{D}\lambda \frac{1}{n} \left( \frac{\pi}{T} \right)^n \sum_{k=1}^n e^{2\pi i k (\lambda^\dagger \lambda + \frac{n}{2} - 1)/n} \\ &\times \exp \left( i \int_0^T d\tau \mathcal{L}_p(\tau) \right), \end{aligned} \quad (8)$$

with

$$\mathcal{L}_p(\tau) = \frac{\dot{x}^2}{2\varepsilon} + \lambda^\dagger \dot{\lambda} + ig \dot{x}_\mu \lambda^\dagger A^\mu \lambda. \quad (9)$$

The important point to note here [12] is that the above Lagrangian, contained in the one loop effective action, is *precisely* the point particle Lagrangian of Balachandran et al., and of Barducci et al. in Eq. (1). This result is extremely suggestive, and is indeed the starting point for our work. The above result in Eqs. (8) and (9) can be easily extended to the full QCD case—the expression for  $\Gamma[A]$  above will also include a path integral over classical spins (see Ref. [11]). Again, this Lagrangian is identical to the generalized Wong Lagrangian [7].

In the following section, we will discuss Wong's equations and show how they follow from Eq. (9).

### 3 Wong's equations and the world line Lagrangian

In recent years, Wong's equations for classical charged particles interacting with classical non-Abelian gauge fields have received a considerable amount of attention in finite temperature applications [25, 26, 31]. Wong wrote down the following set of equations [9]

$$m \frac{dx_I^\mu}{d\tau} = m v_I^\mu = p_I^\mu \quad (10)$$

$$\frac{dp_I^\mu}{d\tau} = g v_\nu^I \text{Tr} (Q_I F^{\mu\nu}(x_I)) \quad (11)$$

$$\dot{Q}_I = -ig [Q_I, v_\mu^I A^\mu] \quad (12)$$

$$D^\nu F_{\nu\mu} = j_\mu \quad (13)$$

and

$$j_\mu(x) = g \sum_{I=1}^K \int d\tau Q_I(\tau) v_\mu^I(\tau) \delta^4[x - x_I(\tau)]. \quad (14)$$

Above,  $\tau$  is the proper time,  $Q^I(\tau) = Q_a^I(\tau) T_a$  is the color charge of the  $I$ -th particle in the adjoint representation, and  $(D_\mu)^{ab} = \delta^{ab} \partial_\mu + g f^{abc} A_\mu^c$  is the

usual covariant derivative. The second equation is the generalization of the Lorentz force equation, the third equation describes color precession. The fourth equation is the Yang–Mills equation, which gives us

$$D_\mu j^\mu = 0 \quad , \quad (15)$$

where  $j^\mu$  is the net color current generated by the classical system of  $K$  particles. A similar set of equations including the spin was also obtained by Heinz [32] starting from the Dirac equation for quarks.

Balachandran et al. showed that the above set of equations can be obtained from the point particle Lagrangian in Eq. (9). We will summarize their results below. The action

$$S = - \int d^4x \frac{1}{4} F_{\mu\nu}^a F^{a,\mu\nu} + \int d\tau \mathcal{L}_p(\tau) , \quad (16)$$

is a) real upto a total time derivative, b) re-parametrization invariant, under  $\tau \rightarrow \tau'$ , and c) invariant under gauge transformations in which *both*  $A_\mu^\alpha$  and  $\lambda_a$  are transformed. Thus under an infinitesimal gauge transformation  $\theta^\alpha$ ,

$$\begin{aligned} A_\mu^\alpha &\longrightarrow (A_\mu^\alpha)^\theta = A_\mu^\alpha - g f_{\alpha\beta\gamma} \theta^\beta A_\mu^\gamma - \partial_\mu \theta^\alpha , \\ \lambda_a &\longrightarrow (\lambda_a)^\theta = \lambda_a + i g \theta^\alpha T_{ab}^\alpha \lambda_b . \end{aligned} \quad (17)$$

Redefining  $\lambda^\dagger \rightarrow i\lambda^\dagger$ , one obtains the equations of motion

$$\dot{\lambda}_a + i g \dot{x}_\mu A_\mu^\alpha T_{ab}^\alpha \lambda_b = 0 , \quad (18)$$

and

$$\frac{\partial}{\partial \tau} \left( \frac{\dot{x}_\mu}{\varepsilon} \right) + g Q^\alpha F_{\mu\nu}^\alpha \dot{x}^\nu = 0 , \quad (19)$$

where

$$Q^\alpha = \lambda_a^\dagger T_{ab}^\alpha \lambda_b . \quad (20)$$

From the preceding equation, and from Eq. (18), we obtain the equation of motion for  $Q^\alpha$  in Eq. (12).

Finally, taking the functional derivative of Eq. (16) with respect to  $A^\mu(x)$ , we obtain the Yang-Mills equations in Eq. (13), with the current  $j^\mu$  defined as in Eq. (14).

In the Hamiltonian formalism,  $x^\mu$ ,  $\lambda_a$  and  $\lambda_a^\dagger$  are independent dynamical variables. Their momentum conjugates are

$$\begin{aligned} P_\mu &= \frac{\dot{x}_\mu}{\varepsilon} - g Q^\alpha A_\mu^\alpha . \\ P_a &= i \lambda_a^\dagger . \\ P_a^\dagger &= 0 . \end{aligned} \quad (21)$$



respectively. Due to the second class constraints, we have to introduce Dirac brackets. For two dynamical variables  $A$  and  $B$  which are even elements of the Grassman algebra we have,

$$\{A, B\}_D = A \left( \frac{\overleftarrow{\partial}}{\partial x^\mu} \frac{\partial}{\partial P^\mu} - \frac{\overleftarrow{\partial}}{\partial P^\mu} \frac{\partial}{\partial x^\mu} - i \frac{\overleftarrow{\partial}}{\partial \lambda_a^\dagger} \frac{\partial}{\partial \lambda_a} - i \frac{\overleftarrow{\partial}}{\partial \lambda_a} \frac{\partial}{\partial \lambda_a^\dagger} \right) B. \quad (22)$$

This gives rise to the following Dirac brackets,

$$\begin{aligned} \{x^\mu, P_\nu\}_D &= \delta_\nu^\mu. \\ \{\lambda_a, \lambda_b^\dagger\}_D &= -i\delta_{ab} \\ \{Q^\alpha, Q^\beta\}_D &= f_{\alpha\beta\gamma} Q^\gamma. \end{aligned} \quad (23)$$

Upon quantization, these give the usual Heisenberg commutation relation for  $P$  and  $x$ , and  $\{\lambda_a, \lambda_b^\dagger\} = \delta_{ab}$ . Also,  $\{Q^\alpha, Q^\beta\}_D$  goes over into the usual commutator. As pointed out by Balachandran et al. and by Barducci et al., the quantized  $\lambda^\dagger$ 's and  $\lambda$ 's act as Fermi creation and annihilation operators, and generate various representations of the color group. The  $n$  creation and annihilation operators span the Clifford algebra  $C_{2n}$ . The sum over  $\phi$ 's in Eq. (7) may therefore be understood as restricting the states generated by  $\lambda$  and  $\lambda^\dagger$  to a particular irreducible representation of the group (see for instance Eq. (3.9) in Ref. [8]).

We have dwelt at some length on the classical and quantum commutation relations among the dynamical variables because they will be useful later on in understanding the transition from the one loop effective action to classical transport theory.

Before we end this section, we should mention that the generalization of the action in Eq. (16) to include classical spins was also performed by Balachandran et al., and by Barducci et al. The classical spins are also Grassmannian variables, and upon quantization, go over into the Dirac  $\gamma$ -matrices. One obtains the generalized Wong equations, including a generalized Bargmann-Michel-Telegdi equation for the time evolution of the Pauli-Lubanski spin four-vector. The generalized point particle Lagrangian is again identical to the one that appears in the one loop effective action for fermions in a classical background field.

## 4 The many body world line formalism

In this section, we will extend the formalism of section 2 to study a many body system of world line scalars. This fills in the step where one goes from a quantum field theoretic description in terms of ensemble averages over fields,

to the classical transport approach where one takes phase space averages over world line trajectories. These averages will of course correspond to the multi-particle distributions of classical transport theory.

In general, for the path integral of a many particle system, one has an ensemble of initial and final conditions<sup>4</sup>(which do not necessarily coincide!). In addition, not all initial and final conditions are weighted equally—the probability for these being given by a density matrix. For example, at finite temperature, in the real time formalism, the initial conditions are specified at a fixed initial time in the complex t-plane, the probability being given by a Boltzmann weight. This is reasonable because in a thermal system operators are of course not “sandwiched” between vacuum states.

We begin again with the full path integral for a massless scalar coupled to an external classical gauge field  $A_\mu$ . The generating functional is

$$\begin{aligned} Z[A_1, A_2] &= \sum_{\psi_f} \langle \psi_f | U(t_{\text{fin}}, t_{\text{init}}) \hat{\rho}_{\text{init}} U(t_{\text{init}}, t_{\text{fin}}) | \psi_f \rangle \\ &= \int_{\mathcal{C}} [d\psi] \exp \left( i \psi^\dagger D^2 \psi \right) \\ &= \int [d\psi_i] [d\psi_f] \mathcal{K}[\psi_i] Z[A_1] Z^*[A_2] \end{aligned} \quad (24)$$

where the subscript  $\mathcal{C}$  represents the usual Schwinger-Keldysh closed time path. One may consider the two pieces of real time paths as representing the two time evolution operators needed to evolve a density operator. Here,  $A_1$  is the background field in the positive time direction,  $A_2$  is the field in the negative time direction and

$$Z[A] = \int_{\psi_i}^{\psi_f} [d\psi_1] \exp \left( i \psi_1^\dagger D^2 \psi_1 \right) \quad (25)$$

The density operator itself is represented here by the matrix element

$$\mathcal{K}[\psi_i] = \langle \psi_i | \hat{\rho}_{\text{init}} | \psi_i \rangle \quad (26)$$

where  $\psi_i$  are eigenvectors diagonalizing the Hermitian matrix  $\rho_{\text{init}}$ .

To rewrite the above in terms of determinants, we note that  $Z[A]$  can be written as

$$Z[A]^{-1} = \prod_n \theta_n \quad (27)$$

where  $\theta_n$  is the eigenvalue of the equation

$$D^2 \psi_n = \theta_n \psi_n \quad (28)$$

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<sup>4</sup>For an excellent reference on path integrals for many particle systems, we recommend the book by Negele and Orland [33].

with the boundary conditions given by  $\psi(t_{\text{init}}) = \psi_i$  and  $\psi(t_{\text{fin}}) = \psi_f$ .

Taking the logarithm of  $Z[A]$  yields

$$\begin{aligned}
-\log Z[A] &= \sum_n \log \theta_n = \sum_n \langle n | \log D^2 | n \rangle \\
&= \sum_n \int d^4x d^4y \sum_{\tilde{\lambda}, \tilde{\lambda}'} \langle n | y, \tilde{\lambda}' \rangle \langle y, \tilde{\lambda}' | \log D^2 | x, \tilde{\lambda} \rangle \langle x, \tilde{\lambda} | n \rangle \\
&= \sum_n \int d^4x d^4y \sum_{\tilde{\lambda}, \tilde{\lambda}'} \psi_n^\dagger(y, \tilde{\lambda}') \langle y, \tilde{\lambda}' | \log D^2 | x, \tilde{\lambda} \rangle \psi_n(x, \tilde{\lambda})
\end{aligned} \tag{29}$$

Here  $\tilde{\lambda}$  is the color label of a particular representation of the algebra<sup>5</sup>. In contrast to the vacuum case, one cannot simply say here that

$$\sum_n \psi_n^\dagger(y, \tilde{\lambda}') \psi_n(x, \tilde{\lambda}) = \delta^{(4)}(x - y) \delta_{\tilde{\lambda}' \tilde{\lambda}}$$

since  $\psi_n$  can only span functions with the same boundary conditions. In the vacuum case, one is able to do this because boundary conditions at  $t = \pm\infty$  are usually trivially the same for any reasonable observables. We now use the identity in Eq. (4), and re-write Eq. (29) using the same line of reasoning as in section 2. (For details of this procedure, see Eqs. (3.16)–(3.22) in the first paper of Ref. [11]). We obtain

$$\begin{aligned}
\Gamma[A, \xi] &= -\log Z[A, \xi] \\
&= \int d^4x d^4y \sum_{\tilde{\lambda}, \tilde{\lambda}'} \xi(x, y, \tilde{\lambda}, \tilde{\lambda}') \int_0^\infty \frac{dT}{T} \mathcal{N} \int_{x, \tilde{\lambda}}^{y, \tilde{\lambda}'} \mathcal{D}z \int_{AP} \mathcal{D}\lambda^\dagger \mathcal{D}\lambda \\
&\times \left( \frac{\pi}{T} \right)^n \frac{1}{n} \sum_{k=1}^n e^{2\pi i k (\lambda^\dagger \lambda + \frac{n}{2} - 1)/n} \exp \left( i \int_0^T d\tau \mathcal{L}_p(z, \lambda, \lambda', A) \right).
\end{aligned} \tag{30}$$

We have replaced in the preceding equation

$$\sum_n \psi_n^\dagger(y, \tilde{\lambda}') \psi_n(x, \tilde{\lambda}) = \xi(x, y, \tilde{\lambda}, \tilde{\lambda}'), \tag{31}$$

where  $\xi$  is a functional of both  $\psi_i$  and  $\psi_f$ . Also,  $\mathcal{L}_p$  is identical to the point particle Lagrangian in Eq. (9).

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<sup>5</sup>As in the earlier sections, the  $\lambda$ 's symbolize the eigenvalues of the coherent state creation and annihilation operators which are generators of arbitrary finite dimensional representations of internal symmetries. We can write  $\psi_n(y, \lambda^\dagger) = \psi_n(x) + \lambda_a^\dagger \psi_{n,a}(x) + \lambda_a^\dagger \lambda_b^\dagger \psi_{n,ab}(x) \cdots$  as in Ref. [7].

Using Eq. (24) and Eq. (30) above, one obtains the following many body path integral,

$$Z = \int [d\xi] \exp(-G[\xi]) \int_{\mathcal{C}} [dA] \exp(iS_{\text{eff}}) , \quad (32)$$

where

$$S_{\text{eff}}[A, \xi] = - \int d^4x \frac{1}{4} F_{\mu\nu}^a F^{a,\mu\nu} + \Gamma[A, \xi] . \quad (33)$$

Here we have made the assumption that the unknown  $\mathcal{K}$  is such that it permits Eq. (32) with a positive definite  $G[\xi]$ . We do know from Eq. (26) that  $\mathcal{K}$  is real since  $\hat{\rho}_{\text{init}}$  is Hermitean. What we do not know is whether the Jacobian associated with the variable change  $(\psi_i, \psi_f) \rightarrow \xi$  is real. This is not clear since it is not guaranteed that  $\xi(x, y, \tilde{\lambda}, \tilde{\lambda}')$  is real. However we will show shortly that when the saddle point approximation is made to obtain classical world lines, only the real part contributes. And in that case, it is likely that the relevant part of  $G[\xi]$  is positive definite.

The many body path integral in Eq. (32) thus includes an average over all possible  $\xi$  with the weight  $e^{-G[\xi]}$ . Since the  $\xi$ 's here are determined exclusively by solving the determinant for the hard modes one arrives at the following physical picture. The above effective action can be thought to represent the dynamics of soft modes which are coupled to a background of hard modes represented by the point particle world lines. That coupling is weighted by a functional  $G[\xi]$ . The above form of the effective action is reminiscent of the small  $x$  effective action [34]. There, by making a separation of scales between small  $x$  and large  $x$ , it was shown that  $G[\xi]$  obeyed a non-linear Wilsonian renormalization group equation [35]. In finite temperature QCD, a similar separation of scales occurs at momenta  $p \sim T, gT, g^2T$ . Thus  $G[\xi] \sim G_{\Lambda}[\xi]$ , where  $\Lambda$  is the scale ( $gT \ll \Lambda \ll T$ ) separating hard and soft modes [17, 37]. It will be interesting to see if renormalization group techniques<sup>6</sup>, analogous to those employed in the small  $x$  effective action, can be used to determine the evolution of  $G[\xi]$  as a function of the cut-off at finite  $T$  [38, 39].

We should comment on the real time contour  $\mathcal{C}$  in the path integral over the effective action in Eq. (32). In the real time formalism of quantum field theory, for thermal initial conditions, the path consists of a piece where the field theory is evolved in imaginary time from  $t = 0$  to  $t = -i\beta$  and two time slices from  $-\infty$  to  $+\infty$  and back from  $\infty - i\beta$  to  $-\infty - i\beta$ . Propagators have a  $2 \times 2$  matrix structure, with the off-diagonal terms corresponding to a flux of on-shell, thermal particles. The corresponding problem of many body propagators in the world line formalism has been addressed by Mathur [40].

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<sup>6</sup>For a discussion of the real time renormalization group, see the book by Goldenfeld [36].

He considers a more general initial state than a thermal one, with the only restriction on the initial density matrix being the constraint that it can be represented as  $\rho = e^{B[\phi]}$ , where  $B$  is quadratic in the field operator  $\phi$ . The thermal distribution is a particular instance of such ‘exponential of quadratic’ density matrices, which have the property that correlators computed with these matrices, satisfy Wick’s theorem.

In the first quantized language of world lines, one obtains the following picture [40]. Since the world line particle moves only in one direction in proper time, the paths backwards in time correspond to switching the sign of the einbein discussed in section 2. Specifying the amplitude  $\xi$  to a reverse orientation of proper time is equivalent to specifying an ‘exponential of quadratic’ density matrix. Replacing  $\xi$  by a  $\delta$ -function, as discussed previously, would let us recover the Feynman propagator. For the particular case of classical transport theory in hot QCD to be discussed in the next section, we will not need to specify the real time contour, but it will be relevant for future detailed computations.

To summarize, in this section, a function  $\xi$  is obtained, and a path integral introduced over all  $\xi$  with an unspecified weight  $G[\xi]$ . Effectively, what one has done is to introduce a density matrix  $\tilde{\rho}$  corresponding to the hard modes in the system. The density matrix of course is required to satisfy the condition

$$\text{Tr}(\tilde{\rho}) = \int [d\xi] \exp(-G[\xi]) = 1. \quad (34)$$

## 5 From the QCD one loop effective action to classical transport theory

In the previous section, we extended the world line formalism discussed in section 2 to a many body context. Our objective in this section is to derive from the one loop effective action in many body QCD under suitable approximations, the classical transport theory of soft classical modes coupled to the world lines of the hard modes. In particular, we will apply the world line formalism to gauge theories at the very high temperatures where weak coupling techniques are applicable. We will see that we recover the coupled set of transport equations that were written down by Litim and Manuel [19]. It was shown by these authors that their results, in turn, agreed with the earlier work, in an apparently very different approach by Bödeker [17].

In hot QCD, the leading high  $T$  contribution to the one loop effective action for gluons is the same as the action for the example we discussed in previous sections—namely, that of a complex scalar field in the adjoint representation. We see this as follows [41]. Working in the background field

gauge, and integrating out the fluctuations to quadratic order in the hard fields, one obtains

$$S_{\text{eff}} = S_{\text{soft}} + \text{Tr} \left( \log(-D^2) \right) - \frac{1}{2} \text{Tr} \left( \log \left[ -D^2 g_{\mu\nu} + 2F_{\mu\nu} \right] \right), \quad (35)$$

where  $D_\mu$  is the usual covariant derivative for the background field. The above equation will also have in general a gauge fixing term-with a gauge fixing parameter  $\alpha$ . For simplicity, we have eliminated this term by setting Feynman gauge,  $\alpha = 1$ . In Ref. [41], it was shown that the  $\alpha$  dependence is suppressed by powers of  $1/T$ .

One can expand the above equation, writing

$$\begin{aligned} S_{\text{eff}} &= S_{\text{soft}} - \text{Tr} \left( \log(-D^2) \right) - \text{Tr} \left( \log \left[ \frac{2F_{\mu\nu}}{D^2} \right] \right) \\ &+ \frac{1}{2} \text{Tr} \left( \frac{1}{D^2} 2F^{\mu\rho} \frac{1}{D^2} 2F^{\rho\nu} \right) + \dots \\ &\sim S_{\text{soft}} + \Gamma[A]. \end{aligned} \quad (36)$$

By power counting arguments [14], the terms containing  $F^{\mu\nu}$  are suppressed by powers of  $T$  at high temperature. Hence,  $\Gamma[A]$  for hot QCD is the same as the  $\Gamma$  we derived in the previous section.

We now wish to derive the classical transport theory that follows from the above effective action. If we insert a functional derivative  $\delta/\delta A$  in Eq. (32), we have the identity

$$\int [d\xi] \exp(-G[\xi]) \int_{\mathcal{C}} [dA] \frac{\delta}{\delta A^\nu} \exp(iS_{\text{eff}}) = 0. \quad (37)$$

This is of course equivalent to the operator equations of motion

$$\langle D_\mu F^{\mu\nu} - J^\nu \rangle \equiv \text{Tr} (\tilde{\rho} (D_\mu F^{\mu\nu} - J^\nu)) = 0. \quad (38)$$

From Eq. (37) and Eq. (30), one obtains

$$\begin{aligned} \langle J^{\nu,\alpha}(x) \rangle &= \int [d\xi] e^{-G[\xi]} \int_{\mathcal{C}} [dA] e^{iS_{\text{eff}}} \int dr dy \sum_{\tilde{\lambda}, \tilde{\lambda}'} \left( \xi(r, y, \tilde{\lambda}, \tilde{\lambda}') \right. \\ &\times \int_0^\infty \frac{dT}{T} \mathcal{N} \int_{r, \tilde{\lambda}}^{y, \tilde{\lambda}'} \mathcal{D}z \int_{AP} \mathcal{D}\lambda^\dagger \mathcal{D}\lambda \left( \frac{\pi}{T} \right)^n \frac{1}{n} \sum_{k=1}^n e^{\frac{2\pi i k}{n} \cdot (\lambda^\dagger \lambda + \frac{n}{2} - 1)} \\ &\times \left. \int_0^T d\tau g \lambda^\dagger T^\beta \lambda \dot{z}_\sigma \frac{\delta A^{\sigma,\beta}(z(\tau))}{\delta A^{\nu,\alpha}(x)} e^{i \int_0^T d\tau' \mathcal{L}_p(z, \lambda, \lambda^\dagger, A)} \right). \end{aligned} \quad (39)$$

Let's now take the saddle point of the Quantum Mechanical path integral in the expectation value of the current<sup>7</sup>. At present, this step can only be justified *a posteriori*. However, it can be anticipated. One expects the world lines of the hard  $p \sim T$  modes to obey classical trajectories *a la Wong* since their coupling to the soft modes is weak at very high temperatures. (This also happens to be the case for the coupling between small  $x$  and large  $x$  modes in the small  $x$  effective action.) This information should of course be obtained self-consistently from the effective action. We hope to address the validity of this classical approximation at a later date.

Taking the saddle point of the Quantum Mechanical path integral, we obtain

$$\langle J^{\mu,\alpha}(x) \rangle = \int [d\xi] \exp(-G[\xi]) \int_{\mathcal{C}} [dA] \exp(iS_{\text{eff}}) J_{\xi,A}^{\mu,\alpha}(x), \quad (40)$$

where

$$\begin{aligned} J_{\xi,A}^{\mu,\alpha}(x) = & \int dr dy \sum_{\tilde{\lambda}, \tilde{\lambda}'} \xi(r, y, \tilde{\lambda}, \tilde{\lambda}') \mathcal{N}' \int_0^\infty \frac{dT}{T} \left( \frac{\pi}{T} \right)^n \frac{1}{n} \\ & \times \exp \left( i \int_0^T d\tau' \mathcal{L}_p(z_c, \lambda_c, \lambda_c^\dagger, A) \right) \\ & \times \int_0^T d\tau g \lambda_c^\dagger T^\alpha \lambda_c \dot{z}_c^\nu \delta^{(4)}(x - z_c(\tau)) \end{aligned} \quad (41)$$

In Eq.(40) the effective action is now defined to be  $S_{\text{eff}} = S_{\text{soft}} + \Gamma_{\text{cl}}$  where

$$\Gamma_{\text{cl}} = \int dr dy \sum_{\tilde{\lambda}, \tilde{\lambda}'} \xi(r, y, \tilde{\lambda}, \tilde{\lambda}') \mathcal{N}' \int_0^\infty \frac{dT}{T} \left( \frac{\pi}{T} \right)^n \exp \left( i \int_0^T d\tau' \mathcal{L}_p(z_c, \lambda_c, \lambda_c^\dagger, A) \right) \quad (42)$$

The subscripts  $c$  on the variables correspond to the classical paths which have the endpoints  $(r, \tilde{\lambda})$  and  $(y, \tilde{\lambda}')$ . Note that for the classical path the  $\tau$  integral is really independent of  $T$  because rescaling  $\tau \rightarrow T\tau$  does not change anything. (Recall from section 3 that the classical point particle Lagrangian is reparametrization invariant.) Hence, extending the range of integration,  $0 < \tau < T \rightarrow 0 < \tau < \infty$  should not modify the result. Performing this

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<sup>7</sup>The sum  $\sum_{k=1}^n e^{\frac{2\pi i k}{n} \cdot (\lambda^\dagger \lambda + n/2 - 1)}$  in the path integral in Eq. (39) is analogous to the well known GSO projection in string theory. As discussed in section 3 (after Eq. (22)), this projection restricts the world line paths to lie in a particular irreducible representation of the group. We will assume implicitly hereafter that this restriction is obeyed by the classical path.

operation results in

$$\begin{aligned}
J_{\xi,A}^{\mu,\alpha}(x) &= \\
& g \int dr dy \sum_{\tilde{\lambda}, \tilde{\lambda}'} \left( \xi(r, y, \tilde{\lambda}, \tilde{\lambda}') \int_0^\infty d\tau \left\{ Q^\alpha(\tau, r, y, \tilde{\lambda}, \tilde{\lambda}') \dot{z}_c^\mu(\tau, r, y, \tilde{\lambda}, \tilde{\lambda}') \right. \right. \\
& \times \left. \left. \delta^{(4)}(x - z(\tau, r, y, \tilde{\lambda}, \tilde{\lambda}')) \right\} \right). \tag{43}
\end{aligned}$$

We have here used Eq. (20) to introduce the color charge  $Q^\alpha$ . Also, we have redefined  $\xi$  to absorb the infinite normalization constant coming from  $\mathcal{N}'$  and the  $T$  integral. A discussion of the renormalization of infinities in the first quantized formalism is outside the scope of this paper. We refer interested readers to the discussion in Ref. [42].

Now introduce the dummy integrals

$$\int dQ \delta^{(n)}(Q - Q(\tau)) = 1 \quad \text{and} \quad \int d^4p \delta^{(4)}(p - p(\tau)) = 1. \tag{44}$$

The color charge measure imposes the constraint (see footnote 7) that the Casimir invariants for the group are conserved. For SU(3) [25],

$$dQ = d^{(8)}Q \delta(Q_a Q^a - q_2) \delta(d_{abc} Q^a Q^b Q^c - q_3), \tag{45}$$

where  $q_2$  and  $q_3$  fix the values of the Casimir invariants, and  $d_{abc}$  are the symmetric constants of the group. The color charges spanning the phase space are dependent variables but, as argued by Kelly et al., they can be formally related to a set of independent phase-space Darboux variables [43]. We will not further address the issue of the color measure in this paper.

Substituting the dummy integrals for  $p$  and  $Q$ , making use of Wong's equations, and switching the order of integration, the current can now be re-written as

$$J_{\xi,A}^{\mu,\alpha}(x) = g \int d^4p dQ p^\mu Q^\alpha f_{\xi,A}(x, p, Q). \tag{46}$$

We have introduced a function  $f_{\xi,A}(x, p, Q)$  which is defined by the relation

$$\begin{aligned}
f_{\xi,A}(x, p, Q) &= \int_0^\infty d\tau \int dr dy \sum_{\tilde{\lambda}, \tilde{\lambda}'} \left( \xi(r, y, \tilde{\lambda}, \tilde{\lambda}') \delta^{(4)}(x - z(\tau, r, y, \tilde{\lambda}, \tilde{\lambda}')) \right. \\
&\times \left. \delta^{(n)}(Q - Q(\tau, r, y, \tilde{\lambda}, \tilde{\lambda}')) \delta^{(4)}(p - p(\tau, r, y, \tilde{\lambda}, \tilde{\lambda}')) \right). \tag{47}
\end{aligned}$$

Consider now what  $f_{\xi,A}$  represents. First note that written in this way, it is clear that only the real part of  $\xi$  contributes to the color current. This can



be easily shown by noticing that  $\xi(r, y, \tilde{\lambda}, \tilde{\lambda}') = \xi^*(y, r, \tilde{\lambda}', \tilde{\lambda})$  and renaming  $r \leftrightarrow y$ ,  $\lambda \leftrightarrow \lambda'$ . Classical paths  $z$  and  $Q$  are not affected by this exchange because given the boundary conditions, these are fixed. The meaning of  $\xi(r, y, \tilde{\lambda}, \tilde{\lambda}')$  is that of the probability density of a particular path that begins with  $(r, \tilde{\lambda})$  and ends with  $(y, \tilde{\lambda}')$ . However, since we are dealing with classical paths here, one can equally well specify a trajectory by the position and the velocity and the value of the charge at any fixed time<sup>8</sup>. (Note that since Eq. (12) is a first order differential equation, one needs only one boundary condition.) In particular, we can do it at  $\tau = 0$ . If we perform the trace explicitly, there will be Jacobian corresponding to the affine transformation. For a discussion of this point, we refer the reader to a paper by Brandt, Frenkel, and Taylor [44]—in particular Eq. (2.14) therein.

Returning to Eq. (47), one can show that  $f_{\xi,A}$  satisfies the equation<sup>9</sup>

$$p^\mu \left( \frac{\partial}{\partial x^\mu} - 2g \text{Tr}(Q F^{\mu\nu}) \frac{\partial}{\partial p^\nu} + 2g \text{Tr}([A_\mu, Q] \frac{\partial}{\partial Q}) \right) f_{\xi,A}(x, p, Q) = 0. \quad (48)$$

Although it appears as such, the above equation is not quite the collisionless Vlasov equation, since  $f_{\xi,A}$  here is a microscopic quantity and not a single particle distribution in the statistical mechanics sense.

To obtain an equation for the single particle distributions, averages both over  $\xi$  and  $A$  must be performed. Keeping in mind that the expressions here are functionals of  $A$  and  $\xi$ , we have

$$\begin{aligned} 0 = p^\mu & \left( \frac{\partial}{\partial x^\mu} \langle f_{\xi,A}(x, p, Q) \rangle - 2g \text{Tr} \left\langle (Q F^{\mu\nu}) \frac{\partial}{\partial p^\nu} f_{\xi,A}(x, p, Q) \right\rangle \right. \\ & \left. - 2g \text{Tr} \left\langle ([Q, A_\mu] \frac{\partial}{\partial Q}) f_{\xi,A}(x, p, Q) \right\rangle \right) \end{aligned} \quad (49)$$

Here the bracket  $\langle \dots \rangle$  indicates the functional averages over  $A$  and  $\xi$  with the weights specified in Eq. (40). The corresponding equation for  $A$  is

$$\langle D_\mu F^{\mu\nu} \rangle = \langle J^\nu \rangle \quad (50)$$

where  $\langle J \rangle$  is expressed in terms of  $f_{\xi,A}$  via Eqns. (40) and (46). Note that the left hand side of this equation consists of 2 and 3 point functions of  $A$ .

The equations (49) and (50) are our main results in their most general form. They are not the Vlasov equations as noted by Litim and Manuel[19]. This is simply because these equations know about 2 and 3 point correlations, whereas in the Vlasov equation only the average values appear. Ignoring these correlations, one would get the Vlasov equation.

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<sup>8</sup>If we do not take the classical saddle point, both boundary conditions will matter.

<sup>9</sup>For an analogous explicit derivation, see Appendix A of Ref. [25]

To consider fluctuations, following Litim and Manuel, we shall now define

$$f_{\xi,A} = \langle f \rangle + \delta f \quad (51)$$

and

$$A = \langle A \rangle + a \quad (52)$$

where  $\langle f \rangle$  and  $\langle A \rangle$  denote the functional average of  $f_{\xi,A}$  and  $A$  in the path integral for the effective action (see Eq. (40)). Note that by definition  $\langle \delta f \rangle$  and  $\langle a \rangle$  both vanish. Furthermore, one has

$$F^{\mu\nu} = \bar{F}^{\mu\nu} + f^{\mu\nu}, \quad (53)$$

where

$$\begin{aligned} \bar{F}^{\mu\nu} &= \partial^\mu \langle A \rangle^{\nu,a} - \partial^\nu \langle A \rangle^{\mu,a} + g f^{abc} \langle A \rangle^{\mu,b} \langle A \rangle^{\nu,c} \\ f^{\mu\nu,a} &= (\bar{D}^\mu a^\nu)^a - (\bar{D}^\nu a^\mu)^a + g f^{abc} a^{\mu,b} a^{\nu,c} \end{aligned} \quad (54)$$

with the covariant derivative  $\bar{D}_\mu$  defined to be  $\partial_\mu - ig \langle A_\mu \rangle$ .

Substituting Eqs. (51) and (52) into Eqs. (49) we obtain

$$p^\mu \left( \bar{D}_\mu \langle f_{\xi,A}(x, p, Q) \rangle - 2g \text{Tr}(Q \bar{F}^{\mu\nu}) \frac{\partial}{\partial p^\nu} \langle f_{\xi,A}(x, p, Q) \rangle \right) = \langle \eta \rangle + \langle \zeta \rangle. \quad (55)$$

In the preceding equation,  $\langle \eta \rangle$  and  $\langle \zeta \rangle$  are given by

$$\begin{aligned} \langle \eta \rangle &= g p^\mu Q^a \langle f_{\mu\nu}^a \partial_p^\nu \delta f \rangle \\ \langle \zeta \rangle &= g Q^a f^{abc} p^\mu \langle a_\mu^b a_\nu^c \rangle \partial_p^\nu \langle f \rangle + g f^{abc} Q^b p^\mu \langle a_\mu^c \partial_{Q^a} \delta f \rangle, \end{aligned} \quad (56)$$

and

$$\bar{D}_\mu \langle f_{\xi,A}(x, p, Q) \rangle = \left( \partial_\mu - g f^{abc} Q^c A_\mu^b \partial_{Q^a} \right) \langle f_{\xi,A}(x, p, Q) \rangle. \quad (57)$$

Performing the same substitution in Eq. (50), we find

$$\bar{D}_\mu \bar{F}^{\mu\nu,a} + \langle J_{\text{fluct}}^{\nu,a} \rangle = \langle J^{\nu,a} \rangle, \quad (58)$$

with  $\langle J_{\text{fluct}}^{\nu,a} \rangle$  defined by the relation

$$\langle J^{\nu,a} \rangle = g f^{abc} \left\langle a_\mu^b \left( ((\bar{D}^\mu a^\nu)^c - (\bar{D}^\nu a^\mu)^c) + g f^{cde} a_d^\mu a_e^\nu \right) + (\bar{D}_\mu a^\mu a^\nu)^{bc} \right\rangle. \quad (59)$$

Proceeding from the preceding set of equations, Litim and Manuel have performed an extensive analysis, in weak coupling, of these equations and their solutions. For instance, neglecting the fluctuation terms introduced

above, and expanding single particle distribution about the equilibrium Bose or Fermi distribution  $\langle f_{\text{equil}} \rangle$ ,

$$\langle f(x, p, Q) \rangle = \langle f_{\text{equil}}^{(0)}(p_0) \rangle + g \langle f^{(1)}(x, p, Q) \rangle + \cdots, \quad (60)$$

one can reconstruct the hard thermal loop effective action, as previously shown by Kelly et al. Keeping the fluctuation terms, and similarly expanding  $\langle \eta \rangle$ ,  $\langle \zeta \rangle$  and  $\langle J_{\text{fluct}} \rangle$ , one obtains the Boltzmann–Langevin equation containing both noise and collision terms that was first derived by Bödeker [17].

We will not perform such an analysis in this section, and wish to direct interested readers to the papers mentioned. The objective of this section was to demonstrate that the starting point for such analyses can be obtained, after clear approximations, from the many body world line formalism developed in this paper. It is hoped that this formalism will allow us to understand better the validity and the limitations of the various approximations employed, and how one may go beyond these approximations.

In particular, it will be useful to better understand the following: a) when does the saddle point approximation for the quantum mechanical path integral break down? Presumably, in this case, the quantum problem can be reformulated in terms of Wigner functions. b) It appears that in this formalism one may be able to distinguish between thermal and quantum fluctuations. How is this achieved in practice? c) The collision-less Vlasov equations can be exponentiated to obtain the hard thermal loop effective action. The addition of damping and noise terms, as Bödeker has emphasized, implies that we are going beyond hard thermal loops. Does consistency then require that one keep the sub-leading terms in Eq. (35)? d) What is the form of the functional  $G[\xi]$ ? Is a renormalization group treatment, wherein  $G$  encapsulates information about non-trivial correlations feasible? We hope to address some of these issues, in the finite temperature context, in a future work [45].

## 6 Discussion

The world line formalism in quantum field theory, going as far back as the early works of Feynman and Schwinger, provides a direct representation of quantum field theory in terms of world lines of point particles. Recently, it has become popular as a powerful method to compute Feynman diagrams in theories with internal symmetries such as spin and color. These internal symmetries are treated using classical Grassmanian variables.

In the many body problem, the world line formalism leads to an intuitive picture consistent with our ideas about classical kinetic theory—that of phase space averages over the world lines of classical particles. We have shown in

this paper that classical transport theory can be obtained from the many body world line formalism. As discussed by several authors, this classical transport theory, in the Vlasov limit sums the leading class of Feynman diagrams for external momenta  $p \sim gT$ —called “hard thermal loops”. The inclusion of noise and damping terms to give the Boltzmann–Langevin equation, goes beyond hard thermal loops and allows one to address physics at the softer scale  $p \sim g^2T$ . The connections of the imaginary time world line formalism to classical transport theory is fleshed out for the specific example of a  $\phi^4$  theory in appendix A.

The potential advantage of the formalism developed in this paper is that one has a formalism at the level of the effective action. It may therefore facilitate computations of effects beyond those of classical transport theory. Its form suggests that it is amenable to a “dynamical renormalization group” treatment, as has been discussed previously by Boyanovsky, de Vega, and collaborators [38]. (See also very recent work, the many body extension of which may lead to a picture similar to ours [46].) Similar ideas have been used previously in small  $x$  physics to develop a Wilsonian renormalization group for wee partons [34, 35]. Indeed, one can draw a formal analogy between the  $P$ ’s and  $Q$ ’s in classical transport theory, and the classical sources in the small  $x$  effective action. Despite the formal similarity, one should caution that the physics in the two cases is quite different. Unlike classical transport theory, small  $x$  physics is dominated by light cone singularities. One consequence is that the light cone sources are static, not dynamic like the  $P$ ’s and the  $Q$ ’s. It is therefore not obvious that the renormalization group ideas of small  $x$  physics can be transported ‘lock, stock and barrel’ over to the physics of hot QCD.

Nevertheless, there are connections that are worth exploring further. A. H. Mueller has shown recently that the classical effective action, when applied to nuclear collisions [47], leads to the Boltzmann equation [48]. Also, recently Elmfors, Hansson, and Zahed have shown that the hard thermal loop free energy could be related directly, via the relativistic virial expansion, to the  $S$ -matrix elements of high energy scattering [41].

We should point out that we have glossed over many subtleties in the world line formalism. In particular, how the closed time path of quantum field theory manifests itself in the first quantized formalism needs to be better understood. Whether one can separate statistical and quantum fluctuations, is another topic that should be studied further. Another issue that needs to be addressed is whether one can go beyond the one loop approximation in the world line formalism. This has been achieved for the vacuum case by Schubert and collaborators [29], and one should be able to extend it in a similar fashion to the many body context. Finally, there are a large number

of physical problems that should be hopefully simpler to compute in the many body world line formalism.

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## A The effective action in the imaginary time world line formalism

In this appendix we derive the leading term in the finite temperature effective action for a  $\phi^4$  theory, using directly the one loop world line formalism at imaginary time.

Starting from the full imaginary time action,

$$S = \int d^4x \left[ -\frac{1}{2}\phi\partial^2\phi + \lambda\phi^4 \right] , \quad (61)$$

where  $\int d^4x = \int_0^\beta d\tau \int d^3x$  and  $\beta$  the inverse temperature  $T$ , we split the field  $\phi$  into a “soft” and “hard” part,  $\phi = \phi_s + \phi_h$ . In the one loop approximation, the action Eq. (61) becomes,

$$S_{1\text{-loop}} = \int d^4x \left[ -\frac{1}{2}\phi_s\partial^2\phi_s + \lambda\phi_s^4 + \frac{1}{2}\phi_h \left( -\partial^2 + 12\lambda\phi_s^2 \right) \phi_h \right] . \quad (62)$$

Integrating out the hard part, we obtain the effective action for the soft field,

$$S_{\text{eff}} = \int d^4x \left[ -\frac{1}{2}\phi_s\partial^2\phi_s + \lambda\phi_s^4 \right] + \frac{1}{2}\text{LogDet} \left( -\partial^2 + 12\lambda\phi_s^2 \right) = S_{\text{eff}}^{(1)} + S_{\text{eff}}^{(2)} . \quad (63)$$

To evaluate the remaining determinant, we use the world line approach at finite temperature (and imaginary time), developed in [49], and write the second term in Eq. (63) as

$$S_{\text{eff}}^{(2)} = -\frac{1}{2} \sum_{n=-\infty}^{\infty} \int_0^{\infty} \frac{dt}{t} \int \mathcal{D}p e^{-\int_0^t dt_1 p^2} \times \int_{P_T} \mathcal{D}x \exp \left[ -\int_0^t dt_1 \left\{ \frac{\dot{x}^2}{4} + 12\lambda \phi_s^2(x(t_1)) \right\} \right]. \quad (64)$$

The boundary condition  $P_T$  in the path integral is given by

$$x_{\mu}(t) = x_{\mu}(0) + n_{\mu} \beta \delta_{\mu 4}, \quad (65)$$

and for simplicity we have put the einbein  $\varepsilon = 2$ .

The term corresponding to the blackbody radiation is easily obtained by taking  $\lambda \rightarrow 0$  and making the substitution

$$x_{\mu}(t_1) = u_{\mu}(t_1) + z_{\mu} + \frac{n_{\mu} \beta t_1}{t} \delta_{\mu 4}, \quad (66)$$

where now  $u_{\mu}(t) = u_{\mu}(0) = 0$  and  $z_{\mu}$  is  $t_1$ -independent. We obtain,

$$S_{\text{eff}}^{(2)}(\lambda = 0) = -\frac{1}{2} \int d^4 z \sum_{n=-\infty}^{\infty} \int_0^{\infty} \frac{dt}{t} e^{-n^2 \beta^2 / 4t} \int \mathcal{D}p e^{-\int_0^t dt_1 p^2} \times \int \mathcal{D}u e^{-\int_0^t dt_1 (\dot{u}^2 / 4)} = \frac{-1}{32\pi^2} \int d^4 z \sum_{n=-\infty}^{\infty} \int_0^{\infty} \frac{dt}{t} e^{-n^2 \beta^2 / 4t}, \quad (67)$$

where we have used the explicit result for a gaussian path integral. The finite temperature part of Eq. (67) is given by

$$S_{\text{eff}}^{(2)}(\lambda = 0) = -\frac{1}{16\pi^2} \int d^4 z \sum_{n=1}^{\infty} \int_0^{\infty} dv v e^{-n^2 \beta^2 v / 4} = -\frac{T^4}{\pi^2} \int d^4 z \sum_{n=1}^{\infty} \frac{1}{n^4} = -\frac{\pi^2 T^4}{90} \int d^4 z, \quad (68)$$

which is just the result for a noninteracting gas of scalar particles. The corresponding result for a free gluon gas is a straightforward generalization of the above procedure, using Eq. (7) to represent the internal degrees of freedom.

To obtain the first nontrivial result, we calculate the corresponding current,

$$j(y) = \frac{\delta S_{\text{eff}}^{(2)}}{\delta \phi_s(y)} = 12\lambda \phi_s(y) \sum_{n=-\infty}^{\infty} \int_0^{\infty} \frac{dt}{t} \int \mathcal{D}p e^{-\int_0^t dt_1 p^2} \int_{P_T} \mathcal{D}x \times$$

$$\exp \left[ - \int_0^t dt_1 \left\{ \frac{\dot{x}^2}{4} + 12\lambda\phi_s^2(x(t_1)) \right\} \right] \left( \int_0^t dt_2 \delta^4(y - x(t_2)) \right), \quad (69)$$

where the delta function is periodic in the imaginary time, with period  $\beta$ . Note that this expression is nothing but the average value of the single particle current, in the background field  $\phi_s$ . Thus, it corresponds to the imaginary time formulation of Eq. (39), although for a  $\phi^4$  theory in the present case.

To lowest order, we may neglect the  $\lambda$ -dependence in the world line action. Using also the substitution in Eq. (66), the current becomes,

$$j(y) \rightarrow j^{(1)}(y) = 12\lambda\phi_s(y) \sum_{n=-\infty}^{\infty} \int_0^{\infty} \frac{dt}{t} e^{-n^2\beta^2/4t} \int \mathcal{D}p e^{-\int_0^t dt_1 p^2} \times \\ \int \mathcal{D}u e^{-\int_0^t dt_1 (\dot{u}^2/4)} \int d^4z \left( \int_0^t dt_2 \delta^4(y - u(t_2) - z - n\beta t_2/t) \right). \quad (70)$$

Now, since

$$\int d^4z \left( \int_0^t dt_2 \delta^4(y - u(t_2) - z - n\beta t_2/t) \right) = \int_0^t dt_2 = t, \quad (71)$$

the current in Eq. (70) becomes

$$j^{(1)}(y) = \frac{3\lambda}{4\pi^2} \phi_s(y) \sum_{n=-\infty}^{\infty} \int_0^{\infty} \frac{dt}{t^2} e^{-n^2\beta^2/4t}, \quad (72)$$

and the finite temperature part is

$$j^{(1)}(y) = \frac{6\lambda T^2}{\pi^2} \phi_s(y) \sum_{n=1}^{\infty} \frac{1}{n^2} = \lambda T^2 \phi_s(y). \quad (73)$$

From the first part of Eq. (63), the blackbody radiation in Eq. (68) and the current in Eq. (73), we finally get the lowest order form of the effective action for the soft field,

$$S_{\text{eff}} = \int d^4z \left[ -\frac{1}{2} \phi_s \partial^2 \phi_s + \lambda \phi_s^4 - \frac{\pi^2 T^4}{90} + \left( \frac{\lambda T^2}{2} \right) \phi_s^2 \right]. \quad (74)$$

The leading term in the effective action corresponds to nothing but the thermal mass for the soft field, as depicted in Fig. 1a. In principle the thermal mass also depends on the infrared cutoff, i.e. the cutoff that characterizes the separation between the hard and soft fields, as discussed in [37]. However, this term is subleading compared to  $T^2$  and has been neglected in the present treatment.

Within the one loop approximation, higher powers of the soft field come from diagrams of the form shown in Fig. 1b, and in general there are of course also corrections from higher loop diagrams.

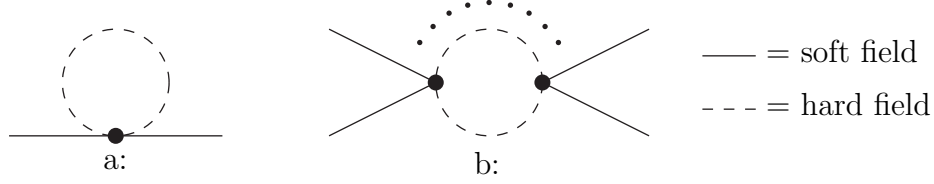


Fig. 1. a: *Thermal mass for the soft field.*  
b: *Higher corrections to the effective action*

Let us now briefly compare the above world line formulation with the classical kinetic theory, as studied in [50, 51]. Within the kinetic theory we would write the current as

$$j_{\text{kin}}(y) = 12\lambda\phi(y) \int dP f(y, p), \quad (75)$$

where the integration measure  $dP$  enforces positivity of the energy and an on-shell constraint,

$$dP = \frac{d^4p}{(2\pi)^3} 2\Theta(p_0)\delta(p_0^2 - \vec{p}^2), \quad (76)$$

and  $f(y, p)$  is a solution to the collisionless Boltzmann equation,

$$\left( \frac{dy_\mu}{d\tau} \frac{\partial}{\partial y_\mu} + \frac{dp_\nu}{d\tau} \frac{\partial}{\partial p_\nu} \right) f(y, p) = (\dot{y} \cdot \partial^{(y)} + \dot{p} \cdot \partial^{(p)}) f(y, p) = 0, \quad (77)$$

with  $\tau$  the proper time. Comparing Eq. (75) with the world line formulation, Eq. (69), we then have to identify

$$\begin{aligned} \int dP f(y, p) &= \langle \int d\tau \delta^4(y - x(\tau)) \rangle = 2 \sum_{n=1}^{\infty} \int_0^{\infty} \frac{dt}{t} \int \mathcal{D}p e^{-\int_0^t dt_1 p^2} \times \\ &\quad \int_{P_T} \mathcal{D}x e^{-\int_0^t dt_1 (x^2/4 + 12\lambda\phi_s^2(x(t_1)))} \left[ \int_0^t dt_2 \delta^4(y - x(t_2)) \right], \end{aligned} \quad (78)$$

where  $P_T$  is the boundary condition given in Eq. (65).

In the weak coupling limit,  $f(y, p)$  can be expanded in a power series in the coupling constant,  $\lambda$ ,

$$f(y, p) = f_0(y, p) + \lambda f_1(y, p) + \mathcal{O}(\lambda^2). \quad (79)$$

To lowest order, the kinetic theory then gives

$$j_{\text{kin}}^{(1)}(y) = 12\lambda\phi(y) \int dP f_0(y, p). \quad (80)$$



With  $f_0$  equal to the equilibrium Bose–Einstein distribution  $n_B = (e^{\beta p_0} - 1)^{-1}$ , this reproduces the lowest order correction from the one–loop effective action, Eq. (73), as expected.

We can also compare the next correction (Fig. 1b with four soft fields), in the limit of zero external four–momentum. Collecting the same powers of  $\lambda$ , by using  $\dot{p}_\mu \sim \lambda \partial_\mu \phi_s^2$  (which also follows from the world line action), the Boltzmann equation gives the solution for  $f_1(y, p)$  as [50, 51, 52],

$$\int dP f_1(y, p) \sim \phi^2(y) \int dP \frac{1}{p_0} \frac{df_0}{dp_0} \sim \phi^2(y) \frac{T}{\Lambda}, \quad (81)$$

and the corresponding term for the current becomes,

$$j_{\text{kin}}^{(2)}(y) \propto \lambda^2 \phi^3(y) \frac{T}{\Lambda}, \quad (82)$$

where  $\Lambda$  is the infrared cutoff on the momentum of the hard particles.

In the world line formulation, instead of a direct infrared cutoff we will assume a mass  $m$  for the scalar field. Neglecting any variation of the external soft field, the one–loop effective action gives a contribution

$$j^{(2)}(y) = -\frac{18\lambda^2 \phi^3(y)}{\pi^2} \sum_{n=1}^{\infty} \int_0^{\infty} \frac{dt}{t} e^{-m^2 t - n^2 \beta^2 / 4t} = -\frac{36\lambda^2 \phi^3(y)}{\pi^2} \sum_{n=1}^{\infty} K_0(m\beta n) \quad (83)$$

with  $K_0(m\beta n)$  the modified Bessel function of the second kind. For  $m\beta \rightarrow 0$ , the leading term comes from

$$\sum_{n=1}^{\infty} K_0(m\beta n) = \frac{\pi T}{2m} + O(\log[m\beta]), \quad (84)$$

which gives a current in agreement with Eq. (82), provided we take  $m \sim \Lambda$ .

This agreement between the two different methods indicates that the conjectured identity in Eq. (78) indeed is true, and gives further evidence for the close relationship between the classical kinetic theory and the one–loop effective action. In particular, Eq. (78) demonstrates how the thermal average of the microscopic one–particle distribution,  $\int d\tau \delta^4(y - x(\tau))$ , naturally appears in the one–loop effective action, when written as a point particle path integral. It also incorporates all the contributions from the soft field, which can then be expanded in a power series in  $\lambda$ , just like  $f(y, p)$ . Thus, it provides an explanation for the apparent success of using a classical transport theory for the hard particles to obtain the effective action for the soft modes.

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